Exam 1 Review

exam format: 8 questions

"hybrid" multiple-choice

→ 25% on answer

→ 75% on work

4" x 6" note card

→ handwritten

→ no sharing

Defective system

$$\vec{x}' = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \vec{x}$$

$$\lambda = 4, 4$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

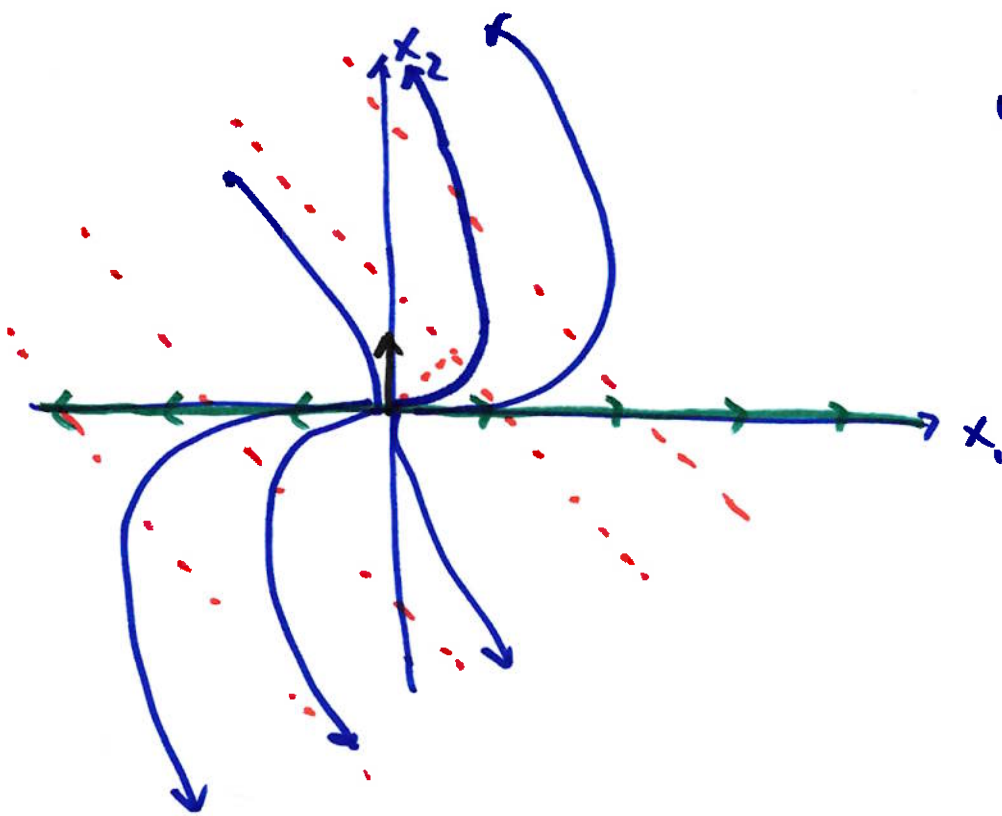
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ missing one eigenvector

generalize eigenvector $\vec{u}$

$$(A - \lambda I)\vec{u} = \vec{v}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$



origin:

unstable nodal source

improper

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follow asymptote

$$\vec{x}' = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix} \vec{x}$$

$$\lambda = 3i, \ -3i$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}, \begin{bmatrix} 2 \\ 1+3i \end{bmatrix}$$

Solution:

$$e^{3it} \begin{bmatrix} 2 \\ 1-3i \end{bmatrix} = e^{i(3t)} \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$$

$$= \cos(3t) + i\sin(3t) \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos(3t) + i\,2\sin(3t) \\ \cos(3t) + 3\sin(3t) + i\sin(3t) - 3i\cos(3t) \end{bmatrix}$$

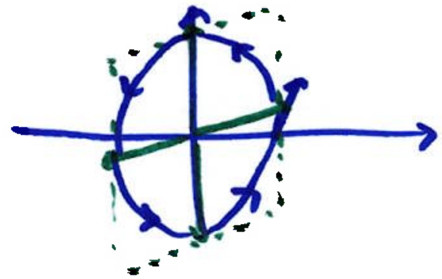
$$= \underbrace{\begin{bmatrix} 2\cos(3t) \\ \cos(3t) + 3\sin(3t) \end{bmatrix}}_{\vec{u}} + i \underbrace{\begin{bmatrix} 2\sin(3t) \\ \sin(3t) - 3\cos(3t) \end{bmatrix}}_{\vec{v}}$$

general solution: $\vec{x} = c_1 \vec{u} + c_2 \vec{v}$

phase portrait: $\lambda = \pm 3i$ origin is center, solutions are ovals

orientation of ovals

$$\begin{bmatrix} 2 \\ 1-3i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$



direction: $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix} \vec{x}$

pick $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

up, right

$x' = x(1-y)$

$y' = y(x-3)$

Critical points: where $x' = 0$ and $y' = 0$

$x = 0, \; y = 1$ (from $x' = 0$)

$y = 0, \; x = 3$ (from $y' = 0$)

cp: $(0,0), \; (3,1)$

linearized system near cp $\rightarrow$ Jacobian matrix

$$x' = f(x,y) \qquad J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$
$$y' = g(x,y)$$

$$x' = x - xy \qquad J = \begin{bmatrix} 1-y & -x \\ y & x-3 \end{bmatrix}$$
$$y' = -3y + xy$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \lambda = 1, -3$$ saddle pt, unstable

$$J(3,1) = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \quad \lambda = \text{imginary}$$ (3,1) is a center, stable

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Competition:
$$x' = x\left(1 - \tfrac{1}{2}x - y\right)$$
$$y' = y\left(2 - \tfrac{1}{5}y - 2x\right)$$

$$f(t) = \begin{cases} 0 & 0 < t \leq \pi \\ \sin(t) & \pi < t < \infty \end{cases}$$

$$= u_\pi(t)\sin(t)$$

$$F(s) = \mathcal{L}\left\{u_\pi(t)\sin(t)\right\}$$

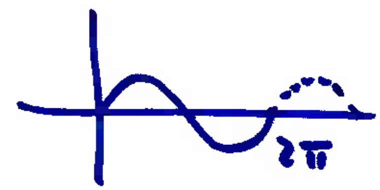
shift left: $t \to t+\pi$

$\sin(a+b)$

$= \sin(a) \cdots$

$$= e^{-\pi s}\,\mathcal{L}\left\{\sin(t+\pi)\right\}$$

$\sin(t+\pi) = -\sin(t)$



$$= e^{-\pi s}\,\mathcal{L}\left\{-\sin(t)\right\}$$

$$= e^{-\pi s} \cdot \frac{-1}{s^2+1}$$

Convolution $\displaystyle\int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = f * g$

$$\mathcal{L}\{f * g\} = FG$$

for example, $\int_0^t \underbrace{(t-\tau)^2}_{f(t)=t^2} \underbrace{\delta(\tau-3)}_{g(t)=\delta(t-3)} d\tau$

Laplace transform is $\frac{2}{s^3} e^{-3s}$

go back to t: $\mathcal{L}^{-1}\left\{ e^{-3s} \frac{2}{s^3} \right\}$

$\frac{2}{s^3} \longrightarrow t^2$ shift RIGHT by 3

$= u_3(t) \cdot (t-3)^2$